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The anticyclic operad of moulds

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Abstract

A new anticyclic operad Mould is introduced, on spaces of functions in several variables. It is proved that the Dendriform operad is an anticyclic suboperad of this operad. Many operations on the free Mould algebra on one generator are introduced and studied. Under some restrictions, a forgetful map from moulds to formal vector fields is then defined. A connection to the theory of tilting modules for quivers of type \mathbb{A} is also described.

0 Introduction

The aim of this article is to build and use a new connection between the theory of operads and the theory of moulds. Operads were introduced in algebraic topology in the 1960's. After being somewhat neglected for some decades, this notion has found a new impetus recently, in connection with mathematical physics, moduli spaces of curves and algebraic combinatorics.

Moulds have a rather different origin. They have been introduced in analysis by J. Ecalle, as a convenient tool to handle complicated singular functions, in relation with his theory of resurgence. Later, he developed around moulds a large apparatus which allowed him to make substantial progress in the theory of polyzetas [Eca02, Eca03, Eca04]. In this article, only the simplest case of moulds will be considered, not the more general case of bimoulds.

The first result of this article is the existence of a very simple structure of operad on moulds, denoted by Mould. In fact, one can define on moulds the finer structure of an anticyclic operad, involving in addition to composition maps some actions of cyclic groups.

This structure is then shown to contain as an anticyclic suboperad the so-called Dendriform operad introduced by J.-L. Loday [Lod01] and denoted by Dend, which has been much studied recently [LR98, LR02]. This provides a radically new point of view on the operad Dend and the key to some new results. The main result is the explicit description of the smallest subset of Dend containing its usual generators and closed under the anticyclic operad structure, by the mean of new combinatorial objects called non-crossing plants.

This article also contains the description of many different operations on moulds, coming either from the operad or from the mould viewpoint, and some of their properties. This is used to prove the existence of a morphism from the Lie algebra of moulds (for one of the Lie brackets and under some restrictions) to the Lie algebra of formal vector fields in one indeterminate. Interesting and

natural examples of moulds are provided and their images by this map are computed.

In some sense, this article provides a reformulation of the basic results of Ecalle on moulds in a more classical algebraic language. This includes notably the so-called ARI bracket, for which we provide a very short definition using the operations obtained from the operad structure. We use this definition to prove some properties of this bracket. It should be said that our setting does not seem to extend to bimoulds, hence can only describe a small part of the theory of Ecalle.

In the last section, it is recalled that the Dendriform operad is strongly related to the theory of tilting modules for the equi-oriented quivers of type \mathbb{A} , and how the results of the present article fit very-well in this relationship. Some conjectural extension of the properties in type \mathbb{A} to other Dynkin diagrams are proposed.

Many thanks to Jean Ecalle for discussions which have led to this research.
MSC 2000: 18D50, 05C05, 05E

1 Notations and definitions

We recall here the terminology we need concerning moulds.

A *mould* is a sequence $(f_n)_{n \geq 1}$, where f_n is a function of the variables $\{u_1, \dots, u_n\}$. A mould is said to have degree n if its only non-zero component is f_n . In this case, this unique component will be denoted f by a convenient abuse of notation.

A mould f of degree n is called *alternal* if it satisfies the following conditions, for $1 \leq i \leq n-1$:

$$\sum_{\sigma \in \text{Sh}(i, n-i)} f(u_{\sigma(1)}, \dots, u_{\sigma(n)}) = 0, \quad (1)$$

where σ runs over the shuffle permutations of $\{u_1, \dots, u_i\}$ and $\{u_{i+1}, \dots, u_n\}$, i.e. permutations such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n)$. One can then extend this definition: a mould is called alternal if each of its components is alternal.

A mould f of degree n is called *vegetal* if it satisfies the following equation:

$$u_1 \dots u_n \sum_{\sigma \in \mathfrak{S}_n} f(t u_{\sigma(1)}, \dots, t u_{\sigma(n)}) = n! f(t, \dots, t), \quad (2)$$

where \mathfrak{S}_n is the permutation group of $\{1, \dots, n\}$.

There is a natural associative product on moulds, defined for f of degree m and g of degree n by

$$\text{MU}(f, g) = f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}). \quad (3)$$

The associated Lie bracket is

$$\text{LIMU}(f, g) = \text{MU}(f, g) - \text{MU}(g, f). \quad (4)$$

We will often use the following convenient shorthand notation:

$$u_{i..j} := \sum_{i \leq k \leq j} u_k. \quad (5)$$

At some places in the text, we will use the following shorthand notation. For a shuffle σ of two ordered sets S' and S'' , let u_σ be the sequence u_s for $s \in S' \cup S''$ in the order specified by σ .

2 The Mould operad

For $n \geq 1$, let $\text{Mould}(n)$ be the vector space of rational functions with rational coefficients in the variables $\{u_1, \dots, u_n\}$. We will show that the collection $\text{Mould} = (\text{Mould}(n))_{n \geq 1}$ has the structure of an anticyclic non-symmetric operad. The reader is referred to [Mar99, MSS02] for the basics of the theory of operads and anticyclic operads.

First, let $\mathbf{1}$ be the function $1/u_1$ in $\text{Mould}(1)$. This will be the unit of the operad.

Let us then introduce a map τ , which is called the *push*. It is defined on $\text{Mould}(n)$ by

$$\tau(f)(u_1, \dots, u_n) = f(-u_{1\dots n}, u_1, \dots, u_{n-1}). \quad (6)$$

Note that τ has order $n+1$ on $\text{Mould}(n)$. It will give the cyclic action of the operad. Let us note also that $\tau(\mathbf{1}) = -\mathbf{1}$. This is one of the axioms of an anticyclic operad.

Let us now introduce the composition maps \circ_i from $\text{Mould}(m) \otimes \text{Mould}(n)$ to $\text{Mould}(m+n-1)$, with $1 \leq i \leq m$. Let f be in $\text{Mould}(m)$ and g be in $\text{Mould}(n)$. The function $f \circ_i g$ is defined by

$$u_{i\dots i+n-1} f(u_1, \dots, u_{i-1}, u_{i\dots i+n-1}, u_{i+n}, \dots, u_{m+n-1}) g(u_i, \dots, u_{i+n-1}). \quad (7)$$

Theorem 2.1 *The push τ and composition maps \circ_i define the structure of an anticyclic non-symmetric operad Mould .*

Proof. One has first to check that these composition maps do indeed define a non-symmetric operad. The unit $\mathbf{1}$ has clearly the expected properties: $\mathbf{1} \circ_1 f = f$ and $f \circ_i \mathbf{1} = f$ for all $f \in \text{Mould}(m)$ and $1 \leq i \leq m$. One has also to check two “associativity” axioms.

Let f, g, h be in $\text{Mould}(m), \text{Mould}(n)$ and $\text{Mould}(p)$.

Let i and j be such that $1 \leq i < j \leq m$. Then one has to check that

$$(f \circ_i g) \circ_{j+n-1} h = (f \circ_j h) \circ_i g. \quad (8)$$

Indeed, both sides are equal to

$$\begin{aligned} & u_{i\dots i+n-1} u_{j+n-1\dots j+n+p-2} g(u_i, \dots, u_{i+n-1}) h(u_{j+n-1}, \dots, u_{j+n+p-2}) \\ & f(u_1, \dots, u_{i-1}, u_{i\dots i+n-1}, u_{i+n}, \\ & \dots, u_{j+n-2}, u_{j+n-1\dots j+n+p-2}, u_{j+n+p-1}, \dots, u_{m+n+p-2}). \end{aligned} \quad (9)$$

Let now i and j be such that $1 \leq i \leq m$ and $1 \leq j \leq n$. One has to check that

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h. \quad (10)$$

Indeed, both sides are equal to

$$\begin{aligned} & f(u_1, \dots, u_{i-1}, u_{i \dots i+p+n-2}, u_{i+p+n-1}, \dots, u_{m+n+p-2}) \\ & \quad g(u_i, \dots, u_{j+i-2}, u_{j+i-1 \dots j+i+p-2}, u_{j+i+p-1}, \dots, u_{i+p+n-2}) \\ & \quad u_{i \dots i+p+n-2} u_{j+i-1 \dots j+i+p-2} h(u_{j+i-1}, \dots, u_{j+i+p-2}). \end{aligned} \quad (11)$$

This proves that Mould is a non-symmetric operad. Then one has to verify that τ gives furthermore an anticyclic structure on this operad.

For this, one has to check two identities. The first one is

$$\tau(f \circ_i g) = \tau(f) \circ_{i-1} g, \quad (12)$$

for $f \in \text{Mould}(m)$, $g \in \text{Mould}(n)$ and $2 \leq i \leq m$. Indeed, this holds true, as both sides are equal to

$$\begin{aligned} & f(-u_{1 \dots m+n-1}, u_1, \dots, u_{i-2}, u_{i-1 \dots n+i-2}, u_{n+i-1}, \dots, u_{m+n-2}) \\ & \quad g(u_{i-1}, \dots, u_{n+i-2}) u_{i-1 \dots n+i-2}. \end{aligned} \quad (13)$$

The other identity that we have to check is

$$\tau(f \circ_1 g) = -\tau(g) \circ_n \tau(f), \quad (14)$$

for $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$. Again, this is true as both sides are equal to

$$-u_{n \dots m+n-1} f(-u_{n \dots m+n-1}, u_n, \dots, u_{m+n-2}) g(-u_{1 \dots m+n-1}, u_1, \dots, u_{n-1}). \quad (15)$$

■

Remark 2.2 It follows from Eq. (7) that if f and g are homogeneous functions of weight d_f and d_g (where all variables u_i are taken with weight 1), then $f \circ_i g$ is also homogeneous of weight $d_f + d_g + 1$. Also, the action of τ clearly preserves the weight in that sense. Hence the collection of subspaces of homogeneous rational functions of weight $-n$ in $\text{Mould}(n)$ is a anticyclic suboperad.

Remark 2.3 Another consequence of Eq. (7) is the following. Let H_n be the product

$$H_n = \prod_{1 \leq i \leq j \leq n} u_{i \dots j}. \quad (16)$$

One can see that, if $H_m f$ and $H_n g$ are polynomials, then so is $H_{m+n-1}(f \circ_i g)$. It is also true that $H_m \tau(f)$ is polynomial if $H_m f$ is, as one can easily check that τ preserves H_m up to sign.

Combining the two previous remarks, one gets that the subspace of $\text{Mould}(n)$ made of homogeneous rational functions f of weight $-n$ such that $H_n f$ is a polynomial define a anticyclic suboperad, which is finite dimensional in each degree.

Remark 2.4 By a similar argument, one can also note that the subspace of rational functions that have only poles of the shape $u_{i \dots j}$ (at some power) for some $i \leq j$ is also stable for the composition and the cyclic action. Such functions will be said to have *nice poles*.

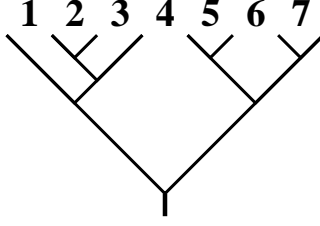


Figure 1: a planar binary tree T and the standard numbering

3 The Dendriform operad

The Dendriform operad was introduced by Loday [Lod01], motivated by some problem in algebraic topology. Later, it was shown to be an anticyclic operad [Cha05a]. We refer the reader to the book [Lod01] for more details on this operad.

Recall that the Dendriform operad is an operad in the category of vector spaces, generated by \curlyvee and \curlywedge of degree 2 with relations

$$\curlyvee \circ_1 (\curlyvee + \curlywedge) = \curlyvee \circ_2 \curlyvee, \quad (17)$$

$$\curlywedge \circ_1 \curlywedge = \curlywedge \circ_2 \curlywedge, \quad (18)$$

$$\curlywedge \circ_1 \curlyvee = \curlywedge \circ_2 (\curlyvee + \curlywedge). \quad (19)$$

The dimension of $\text{Dend}(n)$ is the Catalan number

$$c_n = \frac{1}{n+1} \binom{2n}{n+1}. \quad (20)$$

There is a basis of $\text{Dend}(n)$ indexed by the set $\mathcal{Y}(n)$ of rooted planar binary trees with $n+1$ leaves. In the presentation above, \curlyvee and \curlywedge correspond to the two planar binary trees in $\mathcal{Y}(2)$. We will sometimes denote by Y the unique planar binary tree of degree 1.

The cyclic action is defined on the generators by

$$\tau(\curlyvee) = \curlywedge, \quad (21)$$

$$\tau(\curlywedge) = -(\curlyvee + \curlywedge). \quad (22)$$

Theorem 3.1 *There is a unique map ψ of anticyclic non-symmetric operads from Dend to Mould which maps \curlyvee to $1/(u_1 u_{1\dots 2})$ and \curlywedge to $1/(u_{1\dots 2} u_2)$.*

Proof. It is quite immediate to check, using the known quadratic binary presentation of Dend and the description of the cyclic action on these generators recalled above, that this indeed defines a morphism ψ of operads and that this morphism ψ is a morphism of anticyclic operads. ■

Let us now describe the image by ψ of a planar binary tree T in $\mathcal{Y}(n)$.

Let us define, for each inner vertex v of T , a linear function $\underline{\dim}(v)$ in the variables u_1, \dots, u_n . One can label from left to right the spaces between the leaves from 1 to n as in Fig. 1. Then the vertex v defines a pair of leaves (its leftmost and rightmost descendants), enclosing a subinterval $[i, j]$ of $[1, n]$. Let $\underline{\dim}(v)$ be $u_{i\dots j}$.

Proposition 3.2 *Let T be a planar binary tree. Then its image $\psi(T)$ is the inverse of the product of factors $\underline{\dim}(v)$ over all inner vertices v of T .*

For instance, the image of the tree T of Fig. 1 is

$$\psi(T) = 1 / (u_{1\dots 3}u_{2\dots 3}u_{1\dots 7}u_5u_{5\dots 7}u_7). \quad (23)$$

Proof. The proof is by induction on n . The proposition is true for $n = 1$ or 2 .

Assume that the Proposition is true up to degree n . Let T be a planar binary tree in $\mathcal{Y}(n+1)$. By picking a top vertex of T (any inner vertex of maximal height), one can find a tree S in $\mathcal{Y}(n)$ and an index i such that $T = S \circ_i \vee$ or $T = S \circ_i \wedge$.

Then one can check that the description given above has the correct behavior with respect to such compositions in Mould and in Dend. ■

Let us now introduce a classical map π from permutations of $\{1, \dots, n\}$ to planar binary trees in $\mathcal{Y}(n)$. First, note that one can use the standard numbering as in Fig. 1 to label the inner vertices of a planar binary tree from 1 to n from left to right. Then each tree T induces a natural partial order \leq_T on $\{1, \dots, n\}$ by saying that $i \leq_T j$ if the inner vertex i is below (i.e. is an ancestor of) the inner vertex j . The map π is characterized by the property that $\pi(\sigma) = T$ if and only if the total order $\sigma(1) < \sigma(2) < \dots < \sigma(n)$ is an extension of the partial order \leq_T . In particular, $\sigma(1)$ must be the index of the bottom inner vertex of T . The map π is surjective and has a standard construction by induction, see for example [LR98]. For instance, the images by π of the permutations 4163527 and 4651372 are both the tree of Fig. 1.

Let us define the multi-residue of an element in $\text{Mould}(n)$ according to a permutation $\sigma \in \mathfrak{S}_n$:

$$\oint_{\sigma} f = (2i\pi)^{-n} \oint_{u_{\sigma(1)}} \dots \oint_{u_{\sigma(n)}} f. \quad (24)$$

Proposition 3.3 *For a planar binary tree T in $\mathcal{Y}(n)$ and a permutation $\sigma \in \mathfrak{S}_n$, the multi-residue $\oint_{\sigma} \psi(T)$ does not vanish if and only if $\pi(\sigma) = T$.*

Proof. The proof is by induction. The statement is clear if $n = 1$. Let k be $\sigma(n)$.

Let us assume that $\pi(\sigma) = T$. By the discussion above on the properties of π , this implies that the vertex k is a top vertex of T (a maximal element for \leq_T). Then by computing the innermost residue with respect to u_k , the multi-residue $\oint_{\sigma} \psi(T)$ reduces to the multi-residue of the function obtained by replacing u_k by 0 in $u_k \psi(T)$, with respect to indices $\{1, \dots, n\}$ but k , in some order. By renumbering the variables, this multi-residue is just $\oint_{\sigma'} \psi(T')$ where T' is obtained by removing the top vertex k from T and σ' is the induced permutation of $\{1, \dots, n-1\}$. It is clear that $\pi(\sigma') = T'$, hence the residue $\oint_{\sigma'} \psi(T')$ is not zero by induction and therefore the residue $\oint_{\sigma} \psi(T)$ is not zero too.

Let us now assume that $\pi(\sigma) \neq T$. If the vertex numbered k is not a top vertex, then the residue with respect to u_k is zero, as u_k is not a pole of $\psi(T)$. If the vertex number k is a top vertex, then, with the same notations as above, one necessarily has that $\pi(\sigma') \neq T'$. Hence by induction $\oint_{\sigma'} \psi(T') = 0$ and therefore $\oint_{\sigma} \psi(T) = 0$. ■

Theorem 3.4 *The morphism ψ is injective.*

Proof. It is enough to prove that the functions $\psi(T)$ for all planar binary trees in $\mathcal{Y}(n)$ are linearly independent. This follows from the previous proposition. ■

3.1 The Associative operad

It is known that the Associative operad is the suboperad of the Dendriform operad generated by $\vee + \curlyvee$. Furthermore, the basis of the one-dimensional space $\text{Assoc}(n)$ is mapped in Dend to the sum of all planar binary trees.

Inside the Mould operad, as the image of $\vee + \curlyvee$ is $1/(u_1 u_2)$, one can check by induction that the image of the basis of $\text{Assoc}(n)$ is the inverse of the product $u_1 \dots u_n$.

Remark 3.5 the Associative operad is not stable for the anticyclic structure, though. The smallest anticyclic suboperad of Mould containing Assoc is Dend .

3.2 The graded Tridendriform operad

Several generalizations of the Dendriform operad have been introduced in [Cha02] and [LR04]. They all share the common point that they have in degree n a basis indexed by all planar trees instead of just planar binary trees.

Let us consider among them the operad $gr\text{ TriDend}$ which is the associated graded operad of the Tridendriform operad (which is a filtered operad). It has been considered both in [Cha02] and [LR04] and contains as a suboperad the Dendriform operad. The operad $gr\text{ TriDend}$ is generated by the dendriform generators together with another associative operation \curlyvee in degree 2. One can deduce from the results of [LR04] a presentation by generators and relations of $gr\text{ TriDend}$. It consists of the associativity relations for \curlyvee , of the 3 relations for the dendriform generators \vee and \curlyvee given at the beginning of §3, and of 3 more relations:

$$\curlyvee \circ_1 \vee = \vee \circ_2 \curlyvee, \quad (25)$$

$$\vee \circ_1 \curlyvee = \curlyvee \circ_2 \vee, \quad (26)$$

$$\curlyvee \circ_1 \curlyvee = \curlyvee \circ_2 \curlyvee. \quad (27)$$

Proposition 3.6 *By extending the morphism ψ by $\curlyvee \mapsto \frac{1}{u_1 + u_2}$, one gets a morphism (still denoted by ψ) of operads from $gr\text{ TriDend}$ to Mould.*

Proof. This is an immediate verification. ■

We will not check here whether or not this morphism is injective. This may follow from the same kind of arguments as for the Dendriform operad. Indeed, the image of a planar tree by ψ has a simple description given by an obvious extension of Prop. 3.2.

Remark 3.7 *There is no known cyclic or anticyclic structure on $gr\text{ TriDend}$. The image of $gr\text{ TriDend}$ by ψ is not closed for the action of τ . One may ask for a description of the closure of $gr\text{ TriDend}$ in the anticyclic operad Mould.*

4 Free algebra on one generator

Let us consider the free algebra on one generator for the operad Mould. This can be identified with the direct sum of all spaces Mould(n), which will also be denoted by Mould.

4.1 Dendriform products

The inclusion of the operad Dend in Mould defines the structure of a dendriform algebra on the free Mould algebra on one generator: we have two binary operations \succ and \prec defined for $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$ by

$$f \succ g = \left(\frac{1}{u_1 u_{1\dots 2}} \circ_2 g \right) \circ_1 f = f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{m+1\dots m+n}}{u_{1\dots m+n}} \quad (28)$$

and

$$f \prec g = \left(\frac{1}{u_{1\dots 2} u_2} \circ_2 g \right) \circ_1 f = f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{1\dots m}}{u_{1\dots m+n}}. \quad (29)$$

Proposition 4.1 *If f and g are vegetal, then so are $f \succ g$ and $f \prec g$.*

Proof. Let us do the proof for $f \succ g$, the other case being similar. One has to compute

$$\frac{u_1 \dots u_{m+n}}{u_{1\dots m+n}} \sum_{\sigma \in \mathfrak{S}_{m+n}} f(t u_{\sigma(1)}, \dots, t u_{\sigma(m)}) g(t u_{\sigma(m+1)}, \dots, t u_{\sigma(m+n)}) (u_{\sigma(m+1)} + \dots + u_{\sigma(m+n)}). \quad (30)$$

Let us introduce the set $E = \{\sigma(m+1), \dots, \sigma(m+n)\}$. Then one can rewrite the previous sum as

$$\frac{u_1 \dots u_{m+n}}{u_{1\dots m+n}} \sum_E \left(\sum_{i \in E} u_i \right) \sum_{\sigma} f(t u_{\sigma'(1)}, \dots, t u_{\sigma'(m)}) \sum_{\sigma''} g(t u_{\sigma''(m+1)}, \dots, t u_{\sigma''(m+n)}), \quad (31)$$

where E runs over the set of subsets of cardinal n of $\{1, \dots, m+n\}$, σ' is a bijection from $\{1, \dots, n\}$ to the complement of E and σ'' is a bijection from $\{m+1, \dots, m+n\}$ to E . Using the vegetal property of f and g , this reduces to

$$\frac{m!n!}{u_{1\dots m+n}} f(t, \dots, t) g(t, \dots, t) \sum_E \sum_{i \in E} u_i. \quad (32)$$

Reversing summations, this gives

$$(m+n-1)!n f(t, \dots, t) g(t, \dots, t), \quad (33)$$

which is $(m+n)!(f \succ g)(t, \dots, t)$. Hence $f \succ g$ is vegetal. \blacksquare

4.2 Associative product

The inclusion of the Associative operad in the Mould operad implies that the formula

$$\text{MU}(f, g) = \left(\frac{1}{u_1 u_2} \circ_2 g \right) \circ_1 f \quad (34)$$

defines an associative product on the free Mould-algebra on one generator. One can check that this associative product is exactly the product called MU in the terminology of moulds, see (3). Hence the associated bracket is the so-called LIMU bracket. Note also that $f \succ g + f \prec g = \text{MU}(f, g)$.

As a consequence of Prop. 4.1, one has

Corollary 4.2 *If f and g are vegetal, then so are $\text{MU}(f, g)$ and $\text{LIMU}(f, g)$.*

4.3 Pre-Lie product

Recall that a pre-Lie product on a vector space V is a bilinear map \curvearrowright from V to V such that

$$(x \curvearrowright y) \curvearrowright z - x \curvearrowright (y \curvearrowright z) = (x \curvearrowright z) \curvearrowright y - x \curvearrowright (z \curvearrowright y). \quad (35)$$

This notion is related to manifolds with affine structures and to groups with left-invariant affine structures. As for associative algebras, the corresponding antisymmetric bracket $[x, y] = x \curvearrowright y - y \curvearrowright x$ is a Lie bracket. For a reference on pre-Lie algebras, the reader may consult [CL01].

As there is a injective morphism from the PreLie operad to the symmetric version of the Dendriform operad, hence also to the symmetric version of the Mould operad, one gets a pre-Lie product on the free Mould algebra on one generator and an injective map from the free Pre-Lie algebra on one generator to the free Mould algebra on one generator.

The pre-Lie product is defined by the formula

$$f \curvearrowright g = \left(\frac{1}{u_1 u_{1..2}} \circ_2 f \right) \circ_1 g - \left(\frac{1}{u_{1..2} u_2} \circ_2 g \right) \circ_1 f. \quad (36)$$

or just as $g \succ f - f \prec g$. More explicitly, it is given by

$$\begin{aligned} f(u_1, \dots, u_m) g(u_{m+1}, \dots, u_{m+n}) \frac{u_{m+1..m+n}}{u_{1..m+n}} \\ - g(u_1, \dots, u_n) f(u_{n+1}, \dots, u_{m+n}) \frac{u_{1..n}}{u_{1..m+n}}. \end{aligned} \quad (37)$$

Theorem 4.3 *The pre-Lie product \curvearrowright preserves alternality: if f and g are alternal, then so is $f \curvearrowright g$.*

Proof. Let $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$. Let us fix $j \in \{1, \dots, m+n-1\}$. One has to check that

$$\sum_{\sigma \in \text{Sh}(j, m+n-j)} (f \curvearrowright g)(u_{\sigma(1)}, \dots, u_{\sigma(m+n)}) = 0. \quad (38)$$

Let us compute the first half of this sum:

$$\sum_{\sigma \in \text{Sh}(j, m+n-j)} (g \succ f)(u_{\sigma(1)}, \dots, u_{\sigma(m+n)}). \quad (39)$$

This is

$$\frac{1}{u_{1\dots m+n}} \sum_{\sigma \in \text{Sh}(j, m+n-j)} f(u_{\sigma(1)}, \dots, u_{\sigma(m)}) g(u_{\sigma(m+1)}, \dots, u_{\sigma(m+n)})(u_{\sigma(m+1)} + \dots + u_{\sigma(m+n)}). \quad (40)$$

Let us introduce the set $E' = \{\sigma(1), \dots, \sigma(m)\}$ and let E'' be its complement. Then, by standard properties of shuffles, one can rewrite the previous sum as

$$\frac{1}{u_{1\dots m+n}} \sum_{E'} \sum_{\sigma'} f(u_{\sigma'}) \left(\sum_{k \in E''} u_k \right) \sum_{\sigma''} g(u_{\sigma''}), \quad (41)$$

where E' runs over the set of subsets of $\{1, \dots, m+n\}$ of cardinal m , σ' is a shuffle of $E' \cap \{1, \dots, j\}$ and $E' \cap \{j+1, \dots, m+n\}$ and σ'' is a shuffle of $E'' \cap \{1, \dots, j\}$ and $E'' \cap \{j+1, \dots, m+n\}$. Here we have used the shorthand notation introduced at the end of §1.

Using the alternality of f and g , this sum reduces to the terms where E' is either included in $\{1, \dots, j\}$ or in $\{j+1, \dots, m+n\}$ and E'' is either included in $\{1, \dots, j\}$ or in $\{j+1, \dots, m+n\}$. Hence, the sum reduces to

$$\begin{aligned} \frac{1}{u_{1\dots m+n}} & (f(u_1, \dots, u_j)g(u_{j+1}, \dots, u_{m+n})(u_{j+1} + \dots + u_{m+n}) \\ & + f(u_{j+1}, \dots, u_{m+n})g(u_1, \dots, u_j)(u_1 + \dots + u_j)), \end{aligned} \quad (42)$$

where the first term is present if $j = m$ and the second one if $j = n$.

One can compute in the same way the other half of the sum:

$$\sum_{\sigma \in \text{Sh}(j, m+n-j)} (f \prec g)(u_{\sigma(1)}, \dots, u_{\sigma(m+n)}) \quad (43)$$

and find exactly the same result. Hence the full sum vanishes as expected. \blacksquare

Corollary 4.4 *The image of the free pre-Lie algebra on one generator is contained in the intersection of the free dendriform algebra with set of alternal elements in Mould.*

It seems moreover that this inclusion could be an equality.

Remark 4.5 It follows also from the definition of \prec given above that the Lie bracket LIMU associated to the associative product MU is also the bracket associated to the pre-Lie product \prec .

5 Suboperads in the category of sets

The aim of this section is to describe two small suboperads of the image of Dend in Mould. The key point is that we work here in the category of sets rather than in the category of vector spaces as usual.

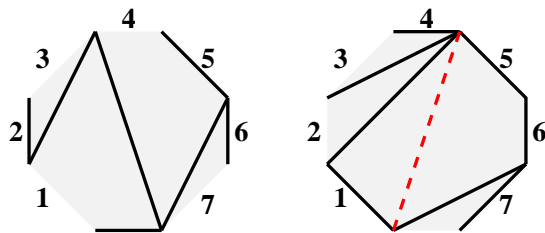


Figure 2: a non-crossing tree and a non-crossing plant

5.1 Combinatorics of non-crossing trees and plants

Let $n \geq 2$ be an integer. Consider the set of vertices of a regular polygon with $n + 1$ sides. One of these sides will be placed at the bottom and called the *base side*. The other sides are then numbered from 1 to n from left to right. A *diagonal* is a line segment drawn between any two vertices of the regular polygon. Two diagonals are *crossing* if they are distinct and meet at some point in the interior of the convex polygon.

An *non-crossing plant* consists of two disjoint subsets of the set of diagonals: the set of *numerator diagonals* (pictured dashed and red) and the set of *denominator diagonals* (pictured plain and black) with the following properties:

- any two diagonals in the union of these subsets are non-crossing,
- the simplicial complex made by the denominator diagonals is connected and contains all vertices,
- any numerator diagonal is contained in a closed cycle of denominator diagonals,
- any closed cycle of denominator diagonals contains exactly one numerator diagonal.

In the sequel, a diagonal will always mean implicitly a denominator diagonal, unless explicitly stated otherwise. Note that a side can only be a denominator diagonal.

Let us call *based non-crossing plant* a non-crossing plant that includes the base side of the regular polygon.

If there is no numerator diagonal, then a non-crossing plant is a *non-crossing tree*, i.e. a maximal set of pairwise non-crossing diagonals whose union is a connected and simply connected simplicial complex (i.e. a tree).

Non-crossing trees and based non-crossing trees are well-known combinatorial objects, see [Noy98] for example. Non-crossing plants seem not to have been considered before. Fig. 2 displays a based non-crossing tree on the left and a non-crossing plant on the right.

We need a precise recursive description of non-crossing plants. One has to distinguish three sorts of them, as depicted in Fig. 3.

The first kind (I) is when the plant is based and the base side is contained in a cycle of denominator diagonals, necessarily of length at least 4, as it must contain a numerator diagonal. For each other diagonal in this cycle, one can

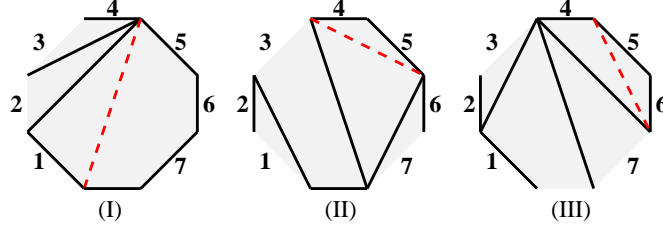


Figure 3: the 3 sorts of non-crossing plants

consider diagonals that are in the connected region (the one not containing the inner part of the cycle) between this diagonal and the boundary of the regular polygon. This defines a based non-crossing plant. Conversely, one can pick any list of length at least 3 of based non-crossing plants and put them on the sides of a closed cycle containing the base side and choose a numerator diagonal in this cycle.

The second kind (II) is when the plant is based, but the base side is not contained in a cycle of diagonals. Then there exists a unique side which is not in the plant and which bounds the same region as the base side. To the left and to the right of the square formed by this side and the base side, one can define two non-crossing plants. This also includes some degenerate cases, where one or both sides are empty. In these cases, the square becomes a triangle or just the base side and there is only one associated non-crossing plant (on the left or on the right) or none.

The third and last kind (III) is when the plant is not based. Consider the unique cycle (of length at least 3) that would be created by adding the base side. Just as in kind (I), one can define, by looking at the outer regions bounded by this cycle, a list of based non-crossing plants of length at least 2.

Let us now translate this trichotomy in terms of generating series and sketch the enumeration of non-crossing plants. We will use the generating series

$$P = \sum_{n \geq 1} p_n x^n = x + 3x^2 + 14x^3 + 80x^4 + \dots, \quad (44)$$

and

$$Q = \sum_{n \geq 1} q_n x^n = x + 2x^2 + 9x^3 + 51x^4 + \dots, \quad (45)$$

where p_n (resp. q_n) is the number of non-crossing plants (resp. based non-crossing plants) in the regular polygon with $n + 1$ sides.

A non-crossing plant is either based or not. If it is not, it is of kind (III) and can be described using a list (of length greater than 2) of based non-crossing plants. Hence one has

$$P = Q + (Q^2/(1 - Q)) = Q/(1 - Q). \quad (46)$$

For a based non-crossing plant, either its base side is contained in a cycle or not. In the first case, it is of kind (I) and can be described using a list (of length k at least 3) of based non-crossing plants and the choice of an inner diagonal in

a cycle of length $k + 1$. In the second case, it is of type (II) and can be described by a pair of non-crossing plants or empty sides. Hence one has

$$Q = \sum_{k \geq 3} \frac{(k+1)(k-2)}{2} Q^k + x(1+P)^2. \quad (47)$$

From these equations, one gets that P satisfies the algebraic relation:

$$x - P + xP^2 + 2xP + P^2 + P^3 = 0. \quad (48)$$

Therefore P has a simple functional inverse:


$$x = \frac{P - P^2 - P^3}{(1+P)^2}. \quad (49)$$

One can remark that this series appear as example (g) in [Lod06].

Let us now introduce the following notions, used in the next section.

A *peeling point* of a non-crossing plant is a vertex (not in the base side) such that the only incident diagonals are sides. In Figure 2, the non-crossing plant on the right has 2 peeling points between sides 3 and 4 and between sides 5 and 6.

Lemma 5.1 *There is always at least one peeling point in a non-crossing plant in the $n + 1$ polygon, for $n \geq 2$.*

Proof. By induction on $n \geq 2$. This is true for all 3 non-crossing plants  in a triangle by inspection. Let us distinguish three cases, as before.

(I) The base side belongs to a cycle of diagonals.

If there is something else than the cycle, there is a peeling point by induction in one of the sub-non-crossing plants bounding the cycle. This gives a peeling point in the whole non-crossing plant.

If there is just a cycle, one can pick any vertex not in the base and not contained in the numerator diagonal. This vertex is a peeling point.

(II) The base side is a diagonal, but does not belong to a cycle of diagonals.

One can consider the left or right sub-non-crossing plant, which contains a peeling point by induction. This provides a peeling point in the full non-crossing plant.

(III) The base side is not a diagonal.

If there is something else than the would-be cycle (see the description of kind (III)), there is a peeling point by induction in one of the sub-non-crossing plants bounding this would-be cycle. This is also a peeling point in the full non-crossing plant.

If there is just a would-be cycle, one can pick as peeling point any vertex not in the base. ■

A *border leaf* is a peeling point in a based non-crossing tree that has only one incident side. In Figure 2, there are 3 border leaves between sides 2 and 3, sides 4 and 5 and sides 6 and 7 in the non-crossing tree on the left.

Lemma 5.2 *There is always at least one border leaf in a based non-crossing tree in the $n + 1$ polygon, for $n \geq 2$.*

Proof. By induction on $n \geq 2$. This is clear if $n = 2$, for the based non-crossing trees \blacktriangleleft and \blacktriangleright . Assume that $n \geq 3$. In any based non-crossing tree, one can define a left and a right subtree, as the connected components of the tree minus its base side (every based non-crossing tree is of kind (II) as a non-crossing plant). At least one of them is not empty. It is enough to prove that there is a border leaf in one of them. One can therefore assume for instance that only the right subtree is not empty.

Then consider the leftmost diagonal (other than the base side) emanating from the right vertex of the base side. Either there is a non-empty non-crossing tree to its right, hence a border leaf inside it by induction, or it has nothing to its right (it is a side). In this case, one can build a new non-crossing tree by shrinking the base side to a point and taking the side to its right as new base side. By induction, this new non-crossing tree has a border leaf. This implies that the initial tree has one too. \blacksquare

5.2 The operad of non-crossing plants

Each diagonal is mapped to a linear function in variables u_1, \dots, u_n as follows: the base side is mapped to $u_{1\dots n}$. The other sides are mapped to u_1, \dots, u_n in the clockwise order. For diagonals which are not sides, one considers the half plane not containing the base side, with respect to this diagonal. This diagonal is mapped to the sum of the values of the sides that are in this half-plane.

To each non-crossing plant, one can then associate a rational function in Mould which is the product of the linear functions associated to its numerator diagonals divided by the product of the linear functions associated to its denominator diagonals. For instance, for the non-crossing tree in the example of Fig. 2, one gets

$$1 / (u_{1\dots 7} u_2 u_{2\dots 3} u_4 \dots 7 u_5 u_6 u_{6\dots 7}), \quad (50)$$

and for the non-crossing plant on the right of the same figure,

$$u_{1\dots 4} / (u_{1\dots 6} u_1 u_{2\dots 4} u_{3\dots 4} u_4 u_5 u_6 u_7). \quad (51)$$

The 3 non-crossing plants in a triangle $\blacktriangleleft, \blacktriangleright$ and \blacktriangle are mapped to $1/(u_1 u_{1\dots 2})$, $1/(u_{1\dots 2} u_2)$ and $1/(u_1 u_2)$ in Mould.

This mapping from the set of non-crossing plants to Mould is obviously injective, as one can recover the non-crossing plant from the factorization of its image. Therefore, we will from now on identify non-crossing plants with their images in Mould.

One can check, using the the definition of the composition in Mould, that the set of non-crossing plants is closed under composition. Let us give a combinatorial description of the composition of non-crossing plants. Given two non-crossing plants f and g in some regular polygons and a side i of the polygon containing f , one has to define a new non-crossing plant in the grafted polygon as in Fig 4. This is simply the union of f and g , with some modification along the grafting diagonal. If this diagonal is present in both f and g , then it is kept in $f \circ_i g$. If it is present in exactly one of f and g , then it is not kept in $f \circ_i g$. If it is present in neither f or g , then it becomes a numerator diagonal in $f \circ_i g$.

Theorem 5.3 *The non-crossing plants form a suboperad NCP in the category of sets, which is contained in the image of Dend. This operad has the following*

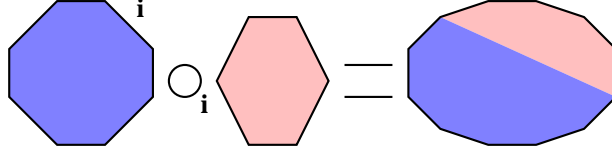


Figure 4: grafting of non-crossing plants: deformation of polygons

presentation: three generators \blacktriangleleft , \blacktriangleright and \blacktriangleup , subject only to the relations:

$$\blacktriangleleft \circ_2 \blacktriangleright = \blacktriangleright \circ_1 \blacktriangleleft, \quad (52)$$

$$\blacktriangleup \circ_1 \blacktriangleleft = \blacktriangleup \circ_2 \blacktriangleleft, \quad (53)$$

$$\blacktriangleleft \circ_1 \blacktriangleup = \blacktriangleleft \circ_2 \blacktriangleup, \quad (54)$$

$$\blacktriangleright \circ_2 \blacktriangleup = \blacktriangleright \circ_1 \blacktriangleup. \quad (55)$$

Proof. The inclusion in the image of Dend will follow from the presentation by generators and relations, with generators \blacktriangleleft , \blacktriangleright and \blacktriangleup that are the images of the elements \vee , \vee and $\vee + \vee$ of Dend in Mould.

There remains to prove the presentation. Let us define an operad NCP' by the presentation above, with \blacktriangleleft , \blacktriangleright and \blacktriangleup replaced by symbols L , R and M . As the relations are satisfied in NCP , there is a unique morphism of operads ∇ from NCP' to NCP sending the generators L , R and M to \blacktriangleleft , \blacktriangleright and \blacktriangleup . Let us prove by induction that there is an inverse Δ to ∇ .

Note that, whenever this makes sense, Δ is of course a morphism of operads. The existence of Δ is clear for $n = 2$. Assume that $n \geq 3$ and let T be in $\text{NCP}(n)$. By Lemma 5.1, there is a peeling point in the non-crossing plant T . Let i be the index of the side of the polygon which is left to this leaf. Then T can be written $S \circ_i \delta$ where δ is either \blacktriangleleft , \blacktriangleright or \blacktriangleup and $S \in \text{NCP}(n-1)$.

Let us define

$$\Delta(T) = \Delta(S) \circ_i \Delta(\delta). \quad (56)$$

One has to prove that this definition does not depend on the choice of the peeling point. Let us assume that there is another peeling point. Without further restriction, one can assume that it is at the right of side j with $i < j$. Thus T can also be written $S' \circ_j \delta'$ where δ' is either \blacktriangleleft , \blacktriangleright or \blacktriangleup and $S' \in \text{NCP}(n-1)$. One then has to distinguish two cases.

Far case

If $i+1 < j$, then there is still a peeling point in S' at the right of edge i , hence there exists $S'' \in \text{NCP}(n-2)$ such that T can be written as

$$S' \circ_j \delta' = (S'' \circ_i \delta) \circ_j \delta' = (S'' \circ_{j-1} \delta') \circ_i \delta = S \circ_i \delta, \quad (57)$$

where the second equality is an axiom of operads. This implies that both choices of peeling point in T leads to the same value for $\Delta(T)$:

$$\begin{aligned} \Delta(S') \circ_j \Delta(\delta') &= (\Delta(S'') \circ_i \Delta(\delta)) \circ_j \Delta(\delta') \\ &= (\Delta(S'') \circ_{j-1} \Delta(\delta')) \circ_i \Delta(\delta) = \Delta(S) \circ_i \Delta(\delta). \end{aligned} \quad (58)$$

Near cases

If $i+1 = j$, then one can distinguish 4 cases. Either all three sides $i, i+1, i+2$ are diagonals, or just two of them are. The other possibilities are excluded by the second condition in the definition of non-crossing plants.

Let us consider the first case. Necessarily T can be written, for some $S'' \in \text{NCP}(n-2)$, as

$$(S'' \circ_i \blacktriangle) \circ_{i+1} \blacktriangle = S'' \circ_i (\blacktriangle \circ_2 \blacktriangle) = S'' \circ_i (\blacktriangle \circ_1 \blacktriangle) = (S'' \circ_i \blacktriangle) \circ_i \blacktriangle. \quad (59)$$

This implies that both choices of peeling point give the same value for $\Delta(T)$:

$$\Delta(S'' \circ_i \blacktriangle) \circ_{i+1} M = (\Delta(S'') \circ_i M) \circ_{i+1} M = (\Delta(S'') \circ_i M) \circ_i M = \Delta(S'' \circ_i \blacktriangle) \circ_i M, \quad (60)$$

where the middle equality follows from relation (53) for the generators L, R and M .

The three other cases are similar to this one, each one of them using one of the relations (52), (54) and (55) for the generators L, R and M .

Hence Δ is well-defined. Then, one has for all T in $\text{NCP}(n)$,

$$\nabla(\Delta(T)) = \nabla(\Delta(S) \circ_i \Delta(\delta)) = S \circ_i \delta = T, \quad (61)$$

by induction hypothesis and because ∇ is a morphism of operad. Let x be in $\text{NCP}'(n)$. Then x can be written $y \circ_i d$ for some y in $\text{NCP}'(n-1)$ and $d \in \{L, R, M\}$. Then

$$\Delta(\nabla(x)) = \Delta(\nabla(y) \circ_i \nabla(d)) = \Delta(\nabla(y)) \circ_i \Delta(\nabla(d)) = y \circ_i d = x. \quad (62)$$

Here for the computation of Δ we choose the peeling point corresponding to $\nabla(d)$. We have proved that Δ is the inverse of ∇ up to order n . This concludes the induction step. ■

Remark 5.4 By adding the opposite of each non-crossing plant, one can get an anticyclic operad in the category of sets. The cyclic action is just given by the rotation of the regular polygon, up to sign.

5.3 The operad of based non-crossing trees

It is not hard to see, using the combinatorial description of the composition given above, that based non-crossing trees are closed under composition.

Proposition 5.5 *The suboperad of NCP generated by \blacktriangleleft and \blacktriangleright is exactly the suboperad of based non-crossing-trees. These generators are only subject to the relation (52).*

Proof. As \blacktriangleleft and \blacktriangleright are based non-crossing trees, the operad they generate is contained in the suboperad of non-crossing trees.

To prove the reverse inclusion, one proceeds by induction. By Lemma 5.2, there is a border leaf in any non-crossing tree T . Let i be the index of the side of the polygon which is left to this leaf of T . Then T can be written $S \circ_i \delta$ where δ is either \blacktriangleleft or \blacktriangleright and S is a smaller based non-crossing tree. This implies the inclusion in the suboperad generated by \blacktriangleleft and \blacktriangleright .

The presentation is a consequence of that of the bigger operad NCP. ■

From results of Loday in [Lod02], one can deduce that

Proposition 5.6 *For any non-crossing plant T of degree n , the inverse image of T in Dend is a sum without multiplicities in $\text{Dend}(n)$ and can therefore be considered as a subset of $\mathcal{Y}(n)$.*

We would like to draw the attention on the following conjecture, which has been checked in low degrees. Recall that the Tamari poset [FT67] is a partial order on the set $\mathcal{Y}(n)$ which indexes a basis of $\text{Dend}(n)$.

Conjecture 5.7 *For any non-crossing tree T of degree n (not necessarily based) in Mould, the inverse image of T in Dend is*

$$\sum_{t \in I} t, \quad (63)$$

where I is some interval in the Tamari poset $\mathcal{Y}(n)$.

6 Other structures

Let us consider some other operations on the free Mould algebra on one generator.

6.1 Over and Under operations

Loday and Ronco have introduced in [LR02] two other associative products on the free dendriform algebra on one generator, called *Over* and *Under* and denoted by $/$ and \backslash . They are usually defined as simple combinatorial operations on planar binary trees, but can be restated using the Dendriform operad as follows:

$$f/g = (g \circ_1 \vee) \circ_1 f \quad (64)$$

and

$$f \backslash g = (f \circ_m \vee) \circ_{m+1} g, \quad (65)$$

where f is assumed to be of degree m . One can use this to extend these operations to the free Mould algebra on one generator. Explicitly, restated inside the Mould operad, these products are given by

$$(f/g)(u_1, \dots, u_{m+n}) = f(u_1, \dots, u_n)g(u_{1\dots n+1}, u_{n+2}, \dots, u_{m+n}) \quad (66)$$

and

$$(f \backslash g)(u_1, \dots, u_{m+n}) = f(u_1, \dots, u_{n-1}, u_{n\dots m+n})g(u_{n+1}, \dots, u_{m+n}). \quad (67)$$

6.2 Structures associated to the operad structure

The fact that Mould is an operad implies that one can define other operations on the free Mould algebra on one generator, namely a pre-Lie product \circ (not to be confused with the one introduced before and denoted by \curvearrowright) and the associated Lie bracket and group law.

The pre-Lie product \circ is defined for $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$ by

$$f \circ g = \sum_{i=1}^m f \circ_i g. \quad (68)$$

More explicitly, $f \circ g$ is given by

$$\sum_{i=1}^m f(u_1, \dots, u_{i-1}, u_{i \dots i+n-1}, u_{i+n}, \dots, u_{m+n-1}) g(u_i, \dots, u_{i+n-1}) u_{i \dots i+n-1}. \quad (69)$$

This construction is clearly functorial from the category of operads to the category of pre-Lie algebras. Note that the product $f \circ g$ is in $\text{Mould}(m+n-1)$.

Theorem 6.1 *The pre-Lie product \circ preserves alternality, that is $f \circ g$ is alternal as soon as f and g are.*

Proof. Let f be in $\text{Mould}(m)$ and $g \in \text{Mould}(n)$. Let us fix $j \in \{1, \dots, m+n-2\}$. One has to check that

$$\sum_{\sigma \in \text{Sh}(j, m+n-1-j)} (f \circ g)(u_{\sigma(1)}, \dots, u_{\sigma(m+n-1)}) = 0. \quad (70)$$

The sum to be computed is

$$\sum_{i=1}^m \sum_{\sigma \in \text{Sh}(j, m+n-1-j)} f(u_{\sigma(1)}, \dots, u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}, \dots, u_{\sigma(m+n-1)}) g(u_{\sigma(i)}, \dots, u_{\sigma(i+n-1)}) (u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}). \quad (71)$$

Let us introduce the sets $E' = \{\sigma(1), \dots, \sigma(i-1)\}$ of cardinal $i-1$ and $E'' = \{\sigma(i+n), \dots, \sigma(m+n-1)\}$ of cardinal $m-i$. They have the following properties:

- $E' \cap \{1, \dots, j\}$ is an initial subset of $\{1, \dots, j\}$,
- $E' \cap \{j+1, \dots, m+n-1\}$ is an initial subset of $\{j+1, \dots, m+n-1\}$,
- $E'' \cap \{1, \dots, j\}$ is a final subset of $\{1, \dots, j\}$,
- $E'' \cap \{j+1, \dots, m+n-1\}$ is a final subset of $\{j+1, \dots, m+n-1\}$.

Let $E = \{\sigma(i), \dots, \sigma(i+n-1)\}$ be the complement of $E' \cup E''$. It follows from the conditions above that $E \cap \{1, \dots, j\}$ and $E \cap \{j+1, \dots, m+n-1\}$ are sub-intervals.

Using standard properties of the set of shuffles, and the shorthand notation u_σ introduced at the end of §1, one can then rewrite the previous sum as

$$\sum_{i=1}^m \sum_{E', E''} \sum_{\sigma', \sigma''} f(u_{\sigma'}, \sum_{k \in E} u_k, u_{\sigma''}) \sum_{\nu} g(u_\nu) \left(\sum_{k \in E} u_k \right), \quad (72)$$

where E' of cardinal $i-1$ and E'' of cardinal $m-i$ are subsets with the above properties, σ' is a shuffle of $E' \cap \{1, \dots, j\}$ and $E' \cap \{j+1, \dots, m+n-1\}$, σ'' is a shuffle of $E'' \cap \{1, \dots, j\}$ and $E'' \cap \{j+1, \dots, m+n-1\}$ and ν is a shuffle of $E \cap \{1, \dots, j\}$ and $E \cap \{j+1, \dots, m+n-1\}$.

Using the alternality of g , one can see that the sum reduces to the cases when $E \subset \{1, \dots, j\}$ or $E \subset \{j+1, \dots, m+n-1\}$. Let us show that each of these two terms vanishes. As the proof is similar, we treat only the case when $E \subset \{1, \dots, j\}$. In this case, there exists k such that $E = \{k, \dots, k+n-1\}$. The corresponding term is

$$\sum_{i=1}^m \sum_{E', E''} \sum_{\sigma', \sigma''} f(u_{\sigma'}, u_{k \dots k+n-1}, u_{\sigma''}) g(u_k, \dots, u_{k+n-1}) u_{k \dots k+n-1}, \quad (73)$$

where E' and E'' runs over the appropriate sets.

Once again by the usual properties of shuffles, this can be rewritten as

$$\sum_k g(u_k, \dots, u_{k+n-1}) (u_{k \dots k+n-1}) \sum_{\mu} f(u_{\mu}), \quad (74)$$

where $1 \leq k \leq k+n-1 \leq j$ and μ is a shuffle of $\{1, \dots, k-1, (k \dots k+n-1), k+n, \dots, j\}$ and $\{j+1, \dots, m+n-1\}$, with the abuse of notation made in considering $(k \dots k+n-1)$ as an index. This sum is zero because f is alternal. ■

Proposition 6.2 *If f and g are vegetal, then so is $f \circ g$.*

Proof. One has to compute

$$\sum_{\sigma \in \mathfrak{S}_{m+n-1}} \sum_{i=1}^m f(t u_{\sigma(1)}, \dots, t(u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}), \dots, t u_{\sigma(m+n-1)}) \\ u_1 \dots u_{m+n-1} g(t u_{\sigma(i)}, \dots, t u_{\sigma(i+n-1)}) t(u_{\sigma(i)} + \dots + u_{\sigma(i+n-1)}). \quad (75)$$

Let us introduce the set $E = \{\sigma(i), \dots, \sigma(i+n-1)\}$. One can then rewrite the previous sum as

$$u_1 \dots u_{m+n-1} t \sum_E \left(\sum_{j \in E} u_j \right) \sum_{i=1}^m \sum_{\sigma'} f(t u_{\sigma'(1)}, \dots, t \left(\sum_{j \in E} u_j \right), \dots, t u_{\sigma'(m+n-1)}) \\ \sum_{\sigma''} g(t u_{\sigma''(i)}, \dots, t u_{\sigma''(i+n-1)}), \quad (76)$$

where E runs over the set of subsets of cardinal n of $\{1, \dots, m+n-1\}$, σ' is a bijection from $\{1, \dots, i-1\} \sqcup \{i+n, \dots, m+n-1\}$ to the complement of E and σ'' is a bijection from $\{i, \dots, i+n-1\}$ to E .

Using first the vegetal property of g , one gets

$$u_1 \dots u_{m+n-1} t \sum_E \frac{\sum_{j \in E} u_j}{\prod_{j \in E} u_j} \sum_{i=1}^m \sum_{\sigma'} f(t u_{\sigma'(1)}, \dots, t \left(\sum_{j \in E} u_j \right), \dots, t u_{\sigma'(m)}) \\ n! g(t, \dots, t). \quad (77)$$

Then, this can be rewritten as

$$n! t g(t, \dots, t) \sum_E \left(\sum_{j \in E} u_j \right) \sum_{\theta} f(t \theta(1), \dots, t \theta(m)) \prod_{j \notin E} u_j. \quad (78)$$

where θ is a bijection from $\{1, \dots, m\}$ to $\{u_j\}_{j \notin E} \sqcup \{\sum_{j \in E} u_j\}$. By using the vegetal property of f , this becomes

$$n! \binom{m+n-1}{n} m! f(t, \dots, t) g(t, \dots, t) t. \quad (79)$$

Using once again the vegetal property of f in the special case $u_1 = n$ and $u_2 = \dots = u_m = 1$, one gets

$$(m+n-1)! \sum_{i=1}^m f(t, \dots, nt, \dots, t) g(t, \dots, t) nt, \quad (80)$$

which is $(m+n-1)!(f \circ g)(t, \dots, t)$. Hence $f \circ g$ is vegetal. \blacksquare

6.3 Forgetful morphism to formal vector fields

The aim of this section is to define a map \mathcal{F} of pre-Lie algebras from moulds satisfying appropriate conditions to formal power series in one variable x , or rather to formal vector fields.

Let us consider here only moulds f such that f_n is homogeneous of weight $-n$ and has only poles at some $u_{i \dots j}$ with arbitrary multiplicity (nice poles). From Remarks 2.2 and 2.4, this subspace is a anticyclic suboperad, hence it is closed for \circ by functoriality. Let us consider its intersection with the subspace of vegetal moulds, which is also closed for \circ by Prop. 6.2. Let us note that this intersection contains the image of Dend by Prop. 4.1 and Prop. 3.2.

Let us recall that the usual pre-Lie product, also denoted by \circ , on vector fields is given by

$$F(x)\partial_x \circ G(x)\partial_x = (\partial_x F(x))G(x)\partial_x. \quad (81)$$

Theorem 6.3 *The substitution $u_i \mapsto 1/x$ induces a morphism \mathcal{F} of pre-Lie algebras $f \mapsto f(x^{-1}, \dots, x^{-1})\partial_x$ from Mould (restricted as above to homogeneous vegetal moulds with nice poles) with the pre-Lie product \circ to the pre-Lie algebra of vector fields in the variable x with formal power series in x as coefficients.*

Proof. Let $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$ satisfying the additional conditions stated before. One has to prove that

$$\begin{aligned} & \partial_x f(x^{-1}, \dots, x^{-1}) g(x^{-1}, \dots, x^{-1}) \\ &= \sum_{i=1}^m n x^{-1} f(x^{-1}, \dots, n x^{-1}, \dots, x^{-1}) g(x^{-1}, \dots, x^{-1}), \end{aligned} \quad (82)$$

where we have used the assumed shape of the poles of f and g to ensure that the substitution makes sense. It is therefore enough to prove that

$$x \partial_x f(x^{-1}, \dots, x^{-1}) = \sum_{i=1}^m n f(x^{-1}, \dots, n x^{-1}, \dots, x^{-1}). \quad (83)$$

From the homogeneity of f , one has to prove that

$$m f(x^{-1}, \dots, x^{-1}) = \sum_{i=1}^m n f(x^{-1}, \dots, n x^{-1}, \dots, x^{-1}), \quad (84)$$

which is a special case of the vegetal property of f , with $t = x^{-1}$, $u_1 = n$ and $u_2 = \dots = u_m = 1$. \blacksquare

As the group law associated to the usual pre-Lie product on formal power series is the classical composition of power series, one can see the group structure on moulds corresponding to \circ as some kind of generalized composition.

Remark 6.4 From (3), it is quite obvious that the associative product MU is mapped by \mathcal{F} to the usual commutative product of formal power series.

6.4 Derivation

Let us introduce a map ∂ on moulds, which decreases the degree by 1.

For a mould $f \in \text{Mould}(m)$, ∂f is the element of $\text{Mould}(m-1)$ defined by

$$\partial f(u_1, \dots, u_{m-1}) = \sum_{j=1}^m \text{Res}_{t=0} f(u_1, \dots, u_{j-1}, t, u_j, \dots, u_{m-1}), \quad (85)$$

where Res is the residue.

The main motivation for this map is the following property.

Proposition 6.5 *The map ∂ is sent by the forgetful map \mathcal{F} to the partial derivative with respect to x , i.e. for any $f \in \text{Mould}(m)$ which is homogeneous, vegetal and has nice poles, one has $\mathcal{F}(\partial f) = \partial_x \mathcal{F}(f)$.*

Proof. Let $f \in \text{Mould}(m)$. By homogeneity, $\mathcal{F}(f) = f(x^{-1}, \dots, x^{-1}) = x^m f(1, \dots, 1)$. Hence $\partial_x \mathcal{F}(f) = m x^{m-1} f(1, \dots, 1)$.

On the other hand, $\mathcal{F}(\partial f)$ is

$$\text{Res}_{t=0} \sum_{j=1}^m f(x^{-1}, \dots, t, \dots, x^{-1}), \quad (86)$$

where t is in the j th position. By the vegetal property of f , this is

$$\text{Res}_{t=0} m \frac{x^{m-1}}{t} f(1, \dots, 1). \quad (87)$$

This proves the expected equality. \blacksquare

Remark 6.6 If f is alternal and of degree at least 2, then $\partial f = 0$. This is obvious once the definition of ∂ is rewritten as the residue of a sum over shuffles of t with $\{u_1, \dots, u_{m-1}\}$.

Proposition 6.7 *The map ∂ is a derivation for the products $\prec, \succ, \curvearrowright$, MU and LIMU. It is also a derivation for \circ , under the restriction that functions have nice poles.*

Proof. It is enough to prove this for \succ and \circ . The case of \prec is similar to the case of \succ and the other cases can be deduced from these ones.

Let us consider the case of \succ . Let $f \in \text{Mould}(m)$ and $g \in \text{Mould}(n)$. One has to compute

$$\begin{aligned} & \sum_{j=1}^m \text{Res}_{t=0} \left(f(u_1, \dots, t, u_j, \dots, u_{m-1}) g(u_m, \dots, u_{m+n-1}) \frac{u_{m\dots m+n-1}}{u_{1\dots m+n-1} + t} \right) + \\ & \sum_{j=m+1}^{m+n} \text{Res}_{t=0} \left(f(u_1, \dots, u_m) g(u_{m+1}, \dots, t, u_j, \dots, u_{m+n-1}) \frac{u_{m+1\dots m+n-1} + t}{u_{1\dots m+n-1} + t} \right). \end{aligned} \quad (88)$$

By the properties of the residue, this becomes

$$\begin{aligned} & \sum_{j=1}^m \text{Res}_{t=0} (f(u_1, \dots, t, u_j, \dots, u_{m-1})) g(u_m, \dots, u_{m+n-1}) \frac{u_{m\dots m+n-1}}{u_{1\dots m+n-1}} + \\ & \sum_{j=m+1}^{m+n} f(u_1, \dots, u_m) \text{Res}_{t=0} (g(u_{m+1}, \dots, t, u_j, \dots, u_{m+n-1})) \frac{u_{m+1\dots m+n-1}}{u_{1\dots m+n-1}}. \end{aligned} \quad (89)$$

This is $\partial f \succ g + f \succ \partial g$, which proves that ∂ is a derivation of \succ .

Let us now consider the case of \circ . One has to compute

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^{i-1} \text{Res}_{t=0} (f(u_1, \dots, t, u_j, \dots, u_{i-2}, u_{i-1\dots i+n-2}, u_{i+n-1}, \dots, u_{m+n-2})) \\ & \quad g(u_{i-1}, \dots, u_{i+n-2}) u_{i-1\dots i+n-2} + \\ & \sum_{i=1}^m \sum_{j=i}^{i+n-1} \text{Res}_{t=0} (f(u_1, \dots, u_{i-1}, u_{i\dots i+n-2} + t, u_{i+n-1}, \dots, u_{m+n-2})) \\ & \quad g(u_i, \dots, t, u_j, \dots, u_{i+n-2}) (u_{i\dots i+n-2} + t) + \\ & \sum_{i=1}^m \sum_{j=i+n}^{m+n-1} \text{Res}_{t=0} (f(u_1, \dots, u_{i-1}, u_{i\dots i+n-1}, u_{i+n}, \dots, t, u_j, \dots, u_{m+n-2})) \\ & \quad g(u_i, \dots, u_{i+n-1}) u_{i\dots i+n-1}. \end{aligned} \quad (90)$$

By the properties of residues, and the assumption that f and g have nice poles, this becomes

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^{i-1} \text{Res}_{t=0} (f(u_1, \dots, t, u_j, \dots, u_{i-2}, u_{i-1\dots i+n-2}, u_{i+n-1}, \dots, u_{m+n-2})) \\ & \quad g(u_{i-1}, \dots, u_{i+n-2}) u_{i-1\dots i+n-2} + \\ & \sum_{i=1}^m \sum_{j=i+n}^{m+n-1} \text{Res}_{t=0} (f(u_1, \dots, u_{i-1}, u_{i\dots i+n-1}, u_{i+n}, \dots, t, u_j, \dots, u_{m+n-2})) \\ & \quad g(u_i, \dots, u_{i+n-1}) u_{i\dots i+n-1} + \\ & \sum_{i=1}^m \sum_{j=i}^{i+n-1} f(u_1, \dots, u_{i-1}, u_{i\dots i+n-2}, u_{i+n-1}, \dots, u_{m+n-2}) \\ & \quad \text{Res}_{t=0} (g(u_i, \dots, t, u_j, \dots, u_{i+n-2})) u_{i\dots i+n-2}. \end{aligned} \quad (91)$$

The first two terms give $\partial f \circ g$ and the third one gives $f \circ \partial g$. ■

As a corollary of Prop. 6.7, the map ∂ preserves the image by ψ of the free Dendriform algebra on one generator. One can be more precise: the action of ∂ is by vertex-removal, in the following sense. From the description of $\psi(T)$ for a planar binary tree T in Prop. 3.2, one can see that taking the residue with respect to one of the variables and then renumbering correspond to the removal of a top vertex in T . Hence $\partial\psi(T)$ is the sum over all top vertices of T of the image by ψ of some smaller binary tree.

6.5 The products ARIT and ARI

Let us define two other bilinear products on the free Mould algebra on one generator, denoted by ARIT and ARI:

$$\text{ARIT}(f, g) = f \circ (g/Y) - f \circ (Y \setminus g) \quad (92)$$

and

$$\text{ARI}(f, g) = \text{ARIT}(f, g) - \text{ARIT}(g, f) + \text{LIMU}(f, g). \quad (93)$$

One can check, by writing the explicit expression for these products, that they do indeed reproduce the ARIT and ARI maps introduced by Ecalle in his study of moulds. In particular, it is known that ARI is a Lie bracket that preserves alternality. Note that we have defined as $\text{ARIT}(f, g)$ what Ecalle denotes by $\text{ARIT}(g)f$.

Lemma 6.8 *There holds $\partial(f/Y) = (\partial f)/Y$ and $\partial(Y \setminus f) = Y \setminus (\partial f)$.*

Proof. Let us consider only the first case, the other one being similar. Let $f \in \text{Mould}(n)$. As

$$f/Y = \varphi \circ_1 f = \frac{1}{u_{1\dots n+1}} f(u_1, \dots, u_n), \quad (94)$$

one has to compute

$$\sum_{i=1}^n \text{Res}_{t=0} \frac{1}{u_{1\dots n} + t} f(u_1, \dots, u_{i-1}, t, u_i, \dots, u_{n-1}) + \text{Res}_{t=0} \frac{1}{u_{1\dots n} + t} f(u_1, \dots, u_n). \quad (95)$$

The second term vanishes, and what remains is

$$\frac{1}{u_{1\dots n}} \sum_{i=1}^n \text{Res}_{t=0} f(u_1, \dots, u_{i-1}, t, u_i, \dots, u_{n-1}), \quad (96)$$

which is exactly $(\partial f)/Y$. ■

Corollary 6.9 *The map ∂ is a derivation for ARIT and ARI.*

Proof. As ∂ is a derivation of \circ by Prop. 6.7, one has

$$\partial(\text{ARIT}(f, g)) = \partial(f) \circ (g/Y) + f \circ \partial(g/Y) - \partial(f) \circ (Y \setminus g) - f \circ \partial(Y \setminus g). \quad (97)$$

One can then conclude for ARIT using Lemma 6.8. The proof for ARI follows from this and from Prop. 6.7. ■

Let us now state some properties of the ARI and ARIT products.

Proposition 6.10 *The free dendriform algebra in Mould is closed under ARIT and ARI. The products ARIT and ARI preserves the vegetal property.*

Proof. For the first statement, it is enough to look at the definition of ARIT. One already knows that the Over and Under operations are defined on the dendriform subspace. The product \circ has the same property, because it is a functorial construction on operads.

Let us prove the second statement. It is enough to prove this for ARIT, by Prop. 4.2. As this property is already known for \circ by Prop. 6.2, it is enough to prove that f/Y and $Y \setminus f$ are vegetal if f is so. Let us consider only the first case, as the other one is just the same.

Let f be in $\text{Mould}(n)$. One has to compute

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} f(u_{\sigma(1)}, \dots, u_{\sigma(n)}) \frac{1}{u_{1\dots n+1} t}. \quad (98)$$

One can separate the sum according to the value of $\sigma(n+1)$, obtaining

$$\frac{1}{u_{1\dots n+1} t} \sum_{i=1}^{n+1} \sum_{\sigma'} f(u_{\sigma'(1)}, \dots, u_{\sigma'(n)}), \quad (99)$$

where σ' runs over the bijections from $\{1, \dots, n\}$ to $\{1, \dots, n+1\} \setminus \{i\}$. Using then the vegetal property of f , this becomes

$$\frac{1}{u_{1\dots n+1} t} \sum_{i=1}^{n+1} \frac{u_i}{u_1 \dots u_{n+1}} n! f(t, \dots, t). \quad (100)$$

This gives

$$\frac{f(t, \dots, t)(n+1)!}{u_1 \dots u_{n+1} (n+1) t}, \quad (101)$$

which proves the expected vegetal property. ■

7 Examples of moulds

Let us describe the image in Mould of some special and nice elements of Dend.

Let AS be the mould defined by $AS_n = 1/(u_1 \dots u_n)$. The components of this mould provide the basis of the Associative suboperad. Hence it is in the image of the Dendriform operad in the Mould operad. On the other hand, it is known that the basis of the Associative suboperad of the Dendriform operad is given by the sum of all planar binary trees. Hence, one has

$$AS_n = \psi \left(\sum_{t \in \mathcal{Y}(n)} T \right) = \frac{1}{u_1 \dots u_n}. \quad (102)$$

One can note that the image by \mathcal{F} of the mould AS is $x/(1-x)$.

Let us say that a planar binary tree is of type (p,q) if its left subtree has $p+1$ leaves and its right subtree has $q+1$ leaves. The sum over binary trees of type (p,q) is

$$\frac{u_p}{u_1 \dots u_n(u_{1\dots n})}. \quad (103)$$

This is an easy consequence of the previous result, using for instance the Over and Under products.

Let TY be the mould defined by

$$TY_n = \frac{\sum_{i=1}^n t^{i-1} u_i}{u_1 \dots u_n u_{1\dots n}}, \quad (104)$$

with a parameter t . By the preceding discussion, the mould TY is also in the image of the Dendriform operad. The image of TY by \mathcal{F} is

$$\frac{1}{1-t} \log \left(\frac{1-tx}{1-x} \right). \quad (105)$$

Another interesting mould has the following components:

$$\frac{\sum_{i=1}^n i u_i}{u_1 \dots u_n u_{1\dots n}}. \quad (106)$$

By the same argument as above, this mould belongs to the image of the Dendriform operad. This mould should be related to the series indexed by planar binary trees considered in [Cha06]. Its image by \mathcal{F} is

$$\frac{x(2-x)}{2(1-x)^2}. \quad (107)$$

One can also compute the image of the Connes-Moscovici series. Let us first recall its definition. In the free pre-Lie algebra on one generator, where the product is denoted by \smile , let CM_1 be the generator and let

$$CM_n = CM_{n-1} \smile CM_1. \quad (108)$$

One can consider these objects as elements of the free dendriform algebra on one generator, endowed with the pre-Lie product \smile . It follows from Prop. 4.3 that these elements are alternal.

Proposition 7.1 *One has*

$$\psi(CM_n) = \frac{1}{u_1 \dots u_n u_{1\dots n}} \sum_{k=1}^n (-1)^{n+k} \binom{n}{k} u_k. \quad (109)$$

The image of $\psi(CM)$ by \mathcal{F} is x .

Proof. The proof is by induction. By definition of the pre-Lie operation \smile in the Dendriform operad, one has

$$CM_{n+1} = \smile \circ_2 CM_n - \smile \circ_1 CM_n. \quad (110)$$

Hence, in Mould, one gets

$$\psi(CM_{n+1}) = \frac{1}{u_1 u_{1\dots 2}} \circ_2 \psi(CM_n) - \frac{1}{u_{1\dots 2} u_2} \circ_1 \psi(CM_n). \quad (111)$$

Explicitly, $\psi(CM_{n+1})$ is given by

$$\frac{u_{2\dots n+1}}{u_1 u_{1\dots n+1}} \psi(CM_n)(u_2, \dots, u_{n+1}) - \frac{u_{1\dots n}}{u_{1\dots n+1} u_{n+1}} \psi(CM_n). \quad (112)$$

Then one can use the addition rule for binomial coefficients and the induction hypothesis. \blacksquare

Another interesting and natural mould PO has the following components:

$$PO_n = \frac{\prod_{i=2}^n u_{1\dots i-1} + t u_i}{u_1 \prod_{i=2}^n (u_i u_{1\dots i})}, \quad (113)$$

with a parameter t . This mould also belongs to the image of the Dendriform operad, as it satisfies the following equation:

$$PO_{n+1} = tPO_n \succ 1/u_1 + PO_n \prec 1/u_1. \quad (114)$$

Obviously, its image by \mathcal{F} is given by the well-known exponential generating series for the Stirling numbers of the first kind:

$$\frac{(1-x)^{-t} - 1}{t}. \quad (115)$$

8 Relation with quivers and tilting modules

There is a nice relationship with the theory of tilting modules for the equi-oriented quivers of type \mathbb{A} (in the classical list of simply-laced Dynkin diagrams). Some properties of this special case may be true in the general case of a Dynkin quiver.

Let Q be the equi-oriented quiver of type \mathbb{A}_n . It is known by a theorem of Gabriel that there is a bijection between indecomposable modules for Q and positive roots for the root system of type \mathbb{A}_n . These positive roots are the sums $\alpha_i + \dots + \alpha_j$ for $1 \leq i \leq j \leq n$, where $\alpha_1, \dots, \alpha_n$ are the simple roots. There is an obvious bijection dim from the set of positive roots to the set of linear functions $u_{i\dots j}$ for $1 \leq i \leq j \leq n$, which is induced by the bijection $\alpha_i \mapsto u_i$.

A tilting module T for the quiver Q is a direct sum of n pairwise non-isomorphic indecomposable modules such that T has no self-extension. One can therefore describe a tilting module T as a set of positive roots, satisfying some condition. Taking the inverse of the product over the corresponding set of linear functions in the variables u , one gets a rational function $\psi(T)$ for each tilting module T . One can check that this set of rational functions is exactly the image in Mould of the set of planar binary trees in Dend by the operad morphism ψ . This gives a natural bijection between tilting modules and planar binary trees.

By this correspondence between tilting modules and trees, the action of the anticyclic rotation τ on the vector space $\text{Dend}(n)$ is mapped to the action induced on the set of roots by the Auslander-Reiten functor on the derived category of the quiver Q .

On the other hand, the action of the anticyclic rotation τ on the vector space $\text{Dend}(n)$ has been related in [Cha05b] to the square of the Auslander-Reiten translation for the derived category of the Tamari poset, which is a classical partial order on the set of planar binary trees. The Tamari poset also has a very natural interpretation in the setting of tilting modules, as a special case of the natural partial order defined by Riedtmann and Schofield [RS91] on the set of tilting modules of a finite-dimensional algebra.

One can try and generalize this to any quiver Q of finite Dynkin type, that is any quiver whose underlying graph is a Dynkin diagram of type \mathbb{A} , \mathbb{D} or \mathbb{E} . The theorem of Gabriel still holds, hence there is a bijection between indecomposable modules and positive roots. One can similarly map positive roots to linear functions in variables u using the decomposition in the basis of simple roots. For an indecomposable module M , the corresponding linear function is

$$\underline{\dim}(M) = \sum_{i=1}^n \dim M_i u_i. \quad (116)$$

There is a finite set of tilting modules for Q . One can, just as above, define a rational function for each tilting module T , as the inverse of the product of the linear functions over the summands of T . For a tilting module $T = \oplus_j M_j$, one gets

$$\psi(T) = 1 / \prod_j \underline{\dim}(M_j) \quad (117)$$

Then, one can ask the following questions:

Question 1: are the functions $\psi(T)$ for all tilting modules T linearly independent ?

Let V_Q be the vector space spanned by the $\psi(T)$ for all tilting modules T .

Question 2: is V_Q stable by the action induced by the action of the Auslander-Reiten translation τ for Q on the set of positive roots ?

Question 3: if so, is this action of the Auslander-Reiten translation for Q related to the Auslander-Reiten translation for the poset of tilting modules for Q defined by Riedtmann and Schofield [RS91, HU05].

Let us note a result in the same spirit, that we have learned from L. Hille [Hil06]: for any Dynkin quiver Q , one has

$$\sum_T \psi(T) = 1/u_1 \dots u_n, \quad (118)$$

where the sum runs over the set of isomorphism classes of tilting modules. This identity comes from a fan related to tilting modules.

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